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REVERSAL EQUATIONS.

BY H. T. MERRITT.

In the September issue of this magazine Prof. W. R. Ransom has called attention in his article on "The Mathematical Pessimist" to a class of equations not recognized in our algebras, and says in this connection: "I consider that in calling attention to the claim of reversal equations to recognition as a *class* of equal importance with linears and quadratics, I have made my chief contribution to algebraic progress."

May I submit that aside from the claim for recognition as a *class,* the reversal equation is worthy of consideration as a *method?* There is a point of view in their solution that is perhaps novel, and certainly helpful in some cases. For several years I have made continual use of this method with review and advanced classes and I am convinced that much can be done with it in beginning classes, although I have never had an opportunity to give it a thorough trial there. The especial value of the method is that it is strongest where the ordinary method is weakest, or at least most troublesome to the beginner; that is, in literal equations and in the transformation of formulas and functional expressions. Anything that will help dispel the phobia that strikes the pupil when he first comes to the requirement, "Solve $A = \frac{1}{2}h(b + b')$ for b," will, I feel sure, be of interest to some teachers. There is here no panacea, but there is something that assures the sufferer that the medicine that he is taking is for a definite purpose.

The reversal method applies to all equations in which the unknown appears only once. This is not so great a restriction as appears, for every equation met in elementary algebra may be reduced to this form. The idea back of the solution of a reversal is to consider what has been done to the unknown in order to *involve* it in the particular manner indicated by the equation, and then by reversing the things done in reverse order, *evolve* the unknown from the equation. The key to the solution is to answer the question, "What has been done to the quantity for which we wish to obtain a value?" For example consider the equation

$$\sqrt{\frac{(3x-4)^3}{2}+5}=3.$$

To solve, the question "What has been done to x?" must be answered first. Note that this cannot be done by pupils whose knowledge of the meaning of the symbols used is at all hazy and that therefore a premium is put on the accurate interpretation of the forms used. Pupils do interpret from the inside outward easily, however, and the answers to this self-asked question are readily and correctly given. In this instance, x is first multiplied by 3, then subtract 4, raise to the third power, divide by 2, add 5 and take the square root, obtaining 3. There now remains the reversal of these operations in the reverse order. We start with the thing obtained, 3, square, subtract 5, multiply by 2, find the cube root, add 4 and divide by 3.

This can be put into compact and convenient form as follows:

$$x = M/3,\ S/4,\ P/3,\ D/2,\ A/5,\ R/2 = 3,$$

$$x = \frac{\sqrt[3]{2(3^2-5)}+4}{3} = 2.$$

As an illustration of a literal, solve for R

$$S = \pi r\sqrt{4R^2 - r^2},$$

$$R = P/2,\ M/4,\ S/r^2,\ R/2,\ M/\pi r = S,$$

$$R = \sqrt{\frac{\left(\frac{S}{\pi r}\right)^2 + r^2}{4}} = \frac{1}{2\pi r}\sqrt{S^2+\pi^2 r^4}.$$

It is in the solution of literal equations of this type and in the transformation of the many formulas of geometry and physics that the pupil will find the reversal method most helpful. The newer texts are laying greater emphasis on the importance of work of this kind, yet the impression prevails among pupils, if not among teachers, that such examples are nothing but a species of algebraic puzzle. I believe that this idea has come from the methods used in solving. The pupil who dares attack an involved expression to solve for a letter contained in it adopts a sort of catch-as-catch-can method and endeavors to pin the shoulders of the unknown to the mat before his own are there. If there is a strategy that he can employ to apparently gain his end, he does not hesitate to adopt it, no matter how doubtful it may be. In the reversal method, however, the plan of attack is definitely laid out. Like Theseus he proceeds against the enemy, laying a trail behind him, in full confidence that the steps can be retraced and that he will come out the victor. The assurance that pupils come to have in their ability to cope successfully with this kind of equation is in itself sufficient ground for considering the method.

Leaving for a little the consideration of the special cases in which the method needs amplification, let us compare it with the method of solving linears commonly used. It will at once be said that in reality there is little new here, as the steps taken to obtain the value are in the end the very steps taken in the usual method. If the reversed steps in the first equation given are performed on both members of the given equation, the equation will then be solved by the usual method. But there is, nevertheless, a great difference to the pupil, as in the reversal method the steps are taken over a return route and are not into a strange territory. There is no better evidence of the difference between the methods than the fact that in the usual method there must be a pause after each step in order to see if the right direction has been maintained. In other words the reversal method has whatever advantages the analytic method of attack has in geometry.

Some six years ago I had occasion to take a class of beginners in algebra and I found early in the work that the class had a most delightfully curious and inquisitive state of mind. They

were not, as a class, at all content to learn a certain operation because it would produce a certain result. I was in perfect sympathy with the text we were using in the effort it made to postpone the word *transposition* and the mechanical part of the process until after the reasons for the step were well understood. But I found that the class was not at all content with the explanation of the change in sign by means of the addition or subtraction axiom. They seemed to get as much satisfaction out of the stock illustrations of the balances, and so on, as most of us do, and they also saw that the addition or subtraction did the work, but still something seemed to be lacking. The better pupils apparently felt that there was a wheel in the train somewhere that was turning, but which they had never seen. Then one day their difficulties vanished as the result of a chance illustration. A long strip of carpet which I had just before had occasion to roll up gave the idea that the quantities of an equation might be compared to objects that would be rolled into the strip. To get out the objects rolled in, the carpet must be unrolled and the objects come out in the reverse order from which they went in. So an unknown which has been *involved,* by any of the six simple operations that the pupil can then command, may be *evolved* by undoing the steps in reverse order. The idea met with an instant response; a few similar illustrations were brought up, such as the blazing of a trail into unknown country, taking an unfamiliar machine apart and putting it together again; and the class had mastered a new idea. This way of looking at the equation satisfied their minds where the axioms did not.

With those excellent principles, the axioms, I have no quarrel, but a boy in one of my classes recently defined them as "something that you say when you can't get out of a thing in any other way," and I suspect his attitude of mind fairly represented that of the class and perhaps reflected in a measure the opinion of his teacher. In the solution of equations the axioms give a good explanation of what has happened after it is all over, but are of no use in telling what to do. The reversal method X-rays the equation and defines the method of attack.

It is obvious that the key question, "What has been done to x?" cannot be answered in either of two cases, one where there

is more than one x and the other where x is an operator. As for the first, the method works after the form of the equation is changed to one with a single x, a step as necessary for the solution by any other method as by the reversal. In the case where x is an operator (the exponentials being not considered) there are two possibilities, x as a divisor and as a subtrahend. The latter is easily disposed of by making the first step a multiplication by -1, as, solve for a

$$s = \frac{rl - a}{r - 1},$$

$$a = M/(-1),\ A/rl,\ D/r - 1 = s,$$

$$a = \frac{(s(r-1) - rl)}{-1} = rl - s(r-1).$$

The case where the unknown is a divisor is the most troublesome, yet even here there may be advantages. The obvious solution is to work with reciprocals, either of the unknown itself, if it occurs as a monomial divisor, or of the polynomial expression in which it occurs as a divisor. An alternative is to place the fraction containing the unknown alone in one member and then take the reciprocal of both members. For example:

$$b = \left(\frac{c}{a - x}\right)^2 + k.$$

Change to

$$\frac{1}{b - k} = \left(\frac{a - x}{c}\right)^2.$$

Then

$$x = M/(-1),\ A/a,\ D/c,\ P/2 = \frac{1}{b - k},$$

$$x = \left[\left(\pm\sqrt{\frac{1}{b-k}}\right) \cdot c - a\right](-1) = \pm c\sqrt{\frac{1}{b-k}} + a.$$

This last example is an interesting one, in that the average pupil may make a bad mess of it by the ordinary method. He will probably expand and clear of fractions as a matter of habit. He will then probably miss the simple trinomial square that would save the day. The solution by formula will then be

chosen, and if the form simplifies, it is a matter for congratulation.

Perhaps the most valuable consideration in connection with the reversal method is that it is more a habit of mind than a method, in its last analysis. Anyone who has a mastery of algebraic processes uses the main points of the method unconsciously, if not consciously. The looking beneath the surface of an expression before doing anything with it is a habit that we try to instil in the early stages of the study. If there is in this method, as here briefly outlined, any suggestion that will help in this direction, it will have served its purpose.

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